The Strong Approximation Conjecture holds for amenable groups

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Abstract.   Let G be a finitely generated group and $G \rhd G_1 \rhd G_2 \rhd \dots$ be normal subgroups such that $\cap_{k=1}^{\infty} G_k = \{1\}$. Let $A \in \mathrm{Mat}_{d \times d}(\mathbb{C}G)$ and $A_k \in \mathrm{Mat}_{d \times d}(\mathbb{C}(G/G_k))$ be the images of A under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück [4]

$$\dim_G(\ker A) = \lim_{k \to \infty} \dim_{G/G_k}(\ker A_k) \,,$$

where $\dim_G$ denotes the von Neumann dimension. In [2] Dodziuk, Linnell, Mathai, Schick and Yates proved the conjecture for torsion free elementary amenable groups. In this paper we extend their result for all amenable groups, using the quasi-tilings of Ornstein and Weiss [6].

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1 Introduction

First, let us recall the approximation result of Dodziuk, Linnell, Mathai, Schick and Yates [2]. Let G be a finitely generated group and let $A \in \mathrm{Mat}_{d \times d}(\mathbb{C}G)$.

Let $l^2(G) = \{ f : G \to \mathbb{C} \mid \sum_{g \in G} |f(g)|^2 < \infty \}$. By left convolution, A induces a bounded linear operator $A : (l^2(G))^d \to (l^2(G))^d$, which commutes with the right G-action. Let

$$\mathrm{proj}_{\ker A} : (l^2(G))^d \to (l^2(G))^d$$

be the orthogonal projection onto ker A. Then

$$\dim_G(\ker A) := \mathrm{Tr}_G(\mathrm{proj}_{\ker A}) := \sum_{i=1}^d \langle \mathrm{proj}_{\ker A} \mathbf{1}_i, \mathbf{1}_i \rangle_{(l^2(G))^d}$$

where $\mathbf{1}_i \in (l^2(G))^d$ is the function which takes the value e_i on the unit element of G and zero elsewhere ($\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{C}^d$). $\dim_G(\ker A)$ is called the von Neumann dimension of ker A.

Now let $G \rhd G_1 \rhd G_2 \rhd \ldots$ be normal subgroups such that $\cap_{k=1}^\infty G_k = \{1\}$.

Let $A_k \in \mathrm{Mat}_{d \times d}(\mathbb{C}(G/G_k))$ be the images of A under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück [4]

$$\dim_G(\ker A) = \lim_{k \to \infty} \dim_{G/G_k}(\ker A_k) \, .$$

In [2] the authors prove the conjecture above in the case when G is a torsion-free elementary amenable group. The goal of this paper is to extend their result to arbitrary amenable groups. If $A \in \mathrm{Mat}_{d \times d}(\mathbb{Z}(G))$ the problem is much easier to handle since one can use the method of Lück [5]. Then the conjecture holds for a large class of groups including amenable and residually finite groups. In the case of complex group algebra the situation seems much more complicated. Dodziuk et al. used noncommutative algebra to prove the conjecture, we shall use the quasi-tilings of Ornstein and Weiss.

2 Preliminaries

Let G be a finitely generated amenable group with a finite symmetric set of generators S. Consider the Cayley-graph C_G, where $V(C_G) = G$ and

$$E(C_G) := \{(x, y) \in G \times G \mid y = sx, s \in S\}.$$

Now we introduce some notation frequently used in the paper later on.

1. If $g \in G$, then its word-length $w(g)$ is defined as $d_{C_G}(g, 1)$, where d_{C_G} is the shortest path distance on the Cayley-graph.

2. Let $F \subset G$ be a finite set, $k > 0$, then $B_k(F)$ denotes the k-neighborhood of F in the d_{C_G}-metric.

3. We denote by $\Omega_k(F)$ the set of vertices p in F, such that $d_{C_G}(p, F^c) > k$, where F^c is the complement of F.

4. For $A \in \mathrm{Mat}_{d \times d}(\mathbb{C}G)$, its propagation $w(A)$ is just $\sup w(g)$, where g runs through the terms of non-zero coefficients in the entries of A. Observe that if $f \in (l^2(G))^d$ and $\mathrm{supp} f = U \subset G$, then $\mathrm{supp}\, A(f) \in B_{w(A)}(U)$, and if $\mathrm{supp} f = \Omega_{w(A)}(U)$ then $\mathrm{supp}\, A(f) \subseteq U$.

5. For a finite set $F \subset G$, ∂F denotes the set of vertices in F such that $d_{C_G}(p, F^c) = 1$. We shall denote the ratio $\frac{|\partial F|}{|F|}$ by $i(F)$.

6. Since G is amenable, it has a *Folner-exhaustion*, that is a sequence of subsets $1 \in F_1 \subset F_2 \subset \ldots, \cup_{n=1}^{\infty} F_n = G$ such that $i(F_n) \to 0$.

   Now we prove some approximation theorems for amenable groups. Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of G and $P_n : (l^2(G))^d \to (l^2(F_n))^d$ be the orthogonal projections. Then by *Theorem* 3.11 [2] (or Proposition 1. [3]) :

$$\dim_G(\ker A) = \lim_{n \to \infty} \frac{\dim_{\mathbb{C}}(\ker P_n A P_n^*)}{|F_n|}.$$

We define the following sequences of vector spaces:

$$Z_n := \{f \in (l^2(G))^d \mid \mathrm{supp} f \subseteq B_{w(A)}(F_n), A(f) \mid_{F_n} = 0\}$$

$$W_n := \{f \in (l^2(G))^d \mid \mathrm{supp} f \subseteq \Omega_{w(A)}(F_n), A(f) = 0\}$$

$$V_n := \ker\left(P_n A P_n^*\right)$$

Proposition 2.1

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(Z_n)}{|F_n|} = \dim_G(\ker A), \quad \lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(W_n)}{|F_n|} = \dim_G(\ker A).$$

Proof. It is enough to prove that

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(Z_n)}{\dim_{\mathbb{C}}(V_n)} = 1 \tag{1}$$

and

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(W_n)}{\dim_{\mathbb{C}}(V_n)} = 1 \tag{2}$$

Clearly, $W_n = V_n \cap \{f \in l^2(F_n)^d \mid \mathrm{supp} f \subseteq \Omega_{w(A)}(F_n)\}$. Hence (2) follows from the fact that

$$\lim_{n\to\infty} \frac{|\Omega_{w(A)}(F_n)|}{|F_n|} = 1 .$$

Also, $W_n = Z_n \cap \{f \in l^2(F_n)^d \mid \mathrm{supp} f \subseteq \Omega_{w(A)}(F_n)\}$. Therefore (1) follows from the fact that

$$\lim_{n\to\infty} \frac{|\Omega_{w(A)}(F_n)|}{|B_{w(A)}(F_n)|} = 1 \qquad \blacksquare$$

Definition 2.1 *Let $F \subset G$ be a finite set and $\delta, \epsilon > 0$ be real numbers. We say that F has property $A(\epsilon, \delta, -)$ if for any subset $K \subseteq F$, $\frac{|K|}{|F|} > 1 - \epsilon$, the following holds:*

- *If $R = \{f \in l^2(F_n)^d \mid \mathrm{supp} f \subseteq \Omega_{w(A)}(K), A(f) = 0\}$, then*

  $$\dim_{\mathbb{C}}(R) \geq (1 - \delta)(\dim_G(\ker A)) .$$

Also, we say that F has property $A(\delta, +)$ if the following holds:

- *If Q is the restriction of the space*

$$Z_F := \{f \in (l^2(G))^d \mid \mathrm{supp} f \subseteq B_{w(A)}(F), A(f) \mid_F = 0\}$$

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onto F, then

$$\dim_{\mathbb{C}}(Q) \leq (\dim_G(\ker A)) + \delta$$

.

Similarly to Propostion 2.1 one can easily prove the following proposition.

Proposition 2.2 *Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of G as above. Then for any pair of real numbers $\delta, \epsilon > 0$ there exists $n_{\delta,\epsilon}$ such that if $n \geq n_{\delta,\epsilon}$ then F_n has both properties $A(\epsilon, \delta, -)$ and $A(\delta, +)$.*

3 Graph convergence and dimension averaging

Let C_G be the Cayley-graph of the previous section. Color the directed edge $(x \to y), x = sy$ by $s \in S$ (hence $(y \to x)$ shall be colored by s^{-1}). Thus we color all edges in both direction with the elements of the set S such a way that for each $x \in G$ the edges outgoing from x are colored in different ways. The following definition is a variation of the one on random weak convergence in [1].

Let $B_1, B_2, \ldots$ be an infinite sequence of finite graphs. Assume that for any $x \in V(B_n)$: $\deg(x) \leq |S|$. We also assume that the directed edges are colored by S such a way that :

- the color of the edge $(x \to y)$ is the inverse of the color of $(y \to x)$.

- the outgoing edges from any vertex are colored differently.

We say that $p \in V(B_n)$ is k-similar to the identity of G, if its k-neighborhood in B_n is edge-colored isomorphic to the k-neighborhood of the identity in C_G. Let Q_k^B be the set of vertices in B that are k-similar to the identity. Then we say that $\{B_n\}_{n=1}^{\infty}$ converge to C_G if for any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $n_{\epsilon,k}$ such that if $n \geq n_{\epsilon,k}$ then

$$Q_k^{B_n} > (1 - \epsilon)|V(B_n)|.$$

Example 1.: Let G be a finitely generated group and $\{B_n\}_{n=1}^{\infty}$ be a sequence of finite induced subgraphs forming a Folner-exhaustion. Then $\{B_n\}_{n=1}^{\infty}$ converge to C_G.

Example 2.: Let G be a finitely generated residually finite group and $G \rhd G_1 \rhd G_2 \rhd \ldots$
be a sequence of finite index normal subgroups such that $\cap_{n=1}^{\infty} G_n = \{1\}$. Let C_n be the
Cayley- graph of G/G_n. Then $\{C_n\}_{n=1}^{\infty}$ converge to C_G.

Now let $A \in \operatorname{Mat}_{d \times d}(\mathbb{C}G)$. One can define the transformation kernel of A, $\widetilde{A} : G \times G \to \operatorname{Mat}_{d \times d}(\mathbb{C})$ the following way. First write A in the form of $\sum_{\gamma \in G} A_\gamma \cdot \gamma$, where
$A_\gamma \in \operatorname{Mat}_{d \times d}(\mathbb{C})$. Then set $\widetilde{A}(\gamma, \delta) := A_{\gamma \delta^{-1}}$. Thus if $f \in l^2(G))^d$, then

$$A(f)(\delta) = \sum_{\gamma \in G} \widetilde{A}(\delta, \gamma) f(\gamma).$$

Now let $\{B_n\}_{n=1}^{\infty}$ be a sequence of graphs converging to C_G. Then we define the finite
dimensional linear transformations $T_n : (l^2(V(B_n))^d \to (l^2(V(B_n))^d$ approximating A, the
following way.

- If $x \in Q_{w(A)}^{B_n}$, $y \in V(B_n)$ and $d_{B_n}(y,x) \leq w(A)$, let $\widetilde{T}_n(y,x) := A(\gamma, 1)$, where γ is the
  element of G satisfying $\phi_{w(A)}^x(\gamma) = y$. Here $\phi_{w(A)}^x$ is the unique colored isomorphism
  between the $w(A)$-neighborhood of 1 in C_G and the $w(A)$-neighborhood of 1 in B_n.

- If $x \notin Q_{w(A)}^{B_n}$ or $d_{B_n}(y,x) > w(A)$, then let $\widetilde{T}_n(x,y) := 0$.

Then if $f \in l^2(V(B_n))^d$ and $p \in V(B_n)$;

$$T_n(f)(p) = \sum_{q \in V(B_n)} \widetilde{T}_n(p,q) f(q).$$

The main goal of our paper is to prove the following theorem.

Theorem 1 *If G is a finitely generated amenable group and $\{B_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$ are as
above, then*

$$\lim_{n \to \infty} \frac{\dim_{\mathbb{C}} \ker T_n}{|V(B_n)|} = \dim_G(\ker A)$$

The Strong Approximation Conjecture for amenable groups follows from the theorem:

Corollary 3.1 *If G is a finitely generated amenable group and*
$G \rhd G_1 \rhd G_2 \ldots, \cap_{n=1}^{\infty} G_n = \{1\}$ *are normal subgroups, then*

$$\lim_{n \to \infty} \dim_{G/G_n}(\ker A_n) = \dim_G(\ker A),$$

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where $A \in Mat_{d \times d}(\mathbb{C}G)$ and $A_n \in Mat_{d \times d}(\mathbb{C}(G/G_n))$ are the images of A under the maps induced by the epimorphisms $G \to G/G_n$.

Proof. (of the Corollary)

<u>Case 1.</u> Suppose that all G_n has finite index. Note that in this case $T_n = A_n$ if n is large enough, hence the corollary immediately follows.

<u>Case 2.</u> Assume that for large enough n, the amenable group G/G_n is infinite. Let $1 \in F_1^n \subset F_2^n \subset \ldots$ be a Folner-exhaustion of the Cayley graph C_{G/G_n} (using the image of the generator system S). Then

$$\dim_{G/G_n}(A_n) = \lim_{k \to \infty} \frac{\dim_{\mathbb{C}}(\ker P_k^n A_n (P_k^n)^*)}{|F_k^n|},$$

where $P_k^n : (l^2(G/G_n))^d \to (l^2(F_k^n))^d$ is the orthogonal projection.

Pick a sequence $F_{m_1}^1, F_{m_2}^2, \ldots$ such that

- $i(F_{m_j}^j) \to 0$.

- $(\dim_{G/G_n}(A_n) - \frac{\dim_{\mathbb{C}}(\ker P_{m_n}^n A_n (P_{m_n}^n)^*)}{|F_{m_n}^n|}) \to 0$.

Now let $B_{m_n}^n$ be the graph induced by $F_{m_n}^n$.

Lemma 3.1 $\{B_{m_n}^n\}_{n=1}^\infty$ *converge to* C_G.

Proof. Since $\cap_{k=1}^\infty G_k = \{1\}$, for any $d \in \mathbb{N}$ there exists $n_d > 0$ such that if $n \geq n_d$ then the d-balls in C_{G/G_n} are colored-isomorphic to the d-ball of C_G. Let $H_{m_n}^n = \Omega_d(F_{m_n}^n)$. Clearly $H_{m_n}^n \subseteq Q_d^{F_{m_n}^n}$. Since the vertex degrees of G/G_n are at most S, $|H_{m_n}^n| \geq |F_{m_n}^n| - |S|^d |\partial F_{m_n}^n|$. Now our lemma easily follows. ∎

Lemma 3.2
$$\lim_{n \to \infty} \frac{\dim_{\mathbb{C}}(\ker P_k^n A_n (P_k^n)^*)}{\dim_{\mathbb{C}} \ker T_n} = 0.$$
Here T_n is the linear operator associated to $B_{m_n}^n$.

If $\text{supp} f \subset F_{m_n}^n \backslash B_{w(A)}(\partial F_{m_n}^n)$ then $T_n(f) = P_k^n A_n (P_k^n)^*(f)$. Since
$$\frac{|F_{m_n}^n \backslash B_{w(A)}(\partial F_{m_n}^n)|}{|F_{m_n}^n|} \to 1$$
our lemma follows. ∎

Obviously, Lemma 3.1 and Lemma 3.2 imply the corollary. ∎

4 Quasi-tilings

Let us recall the notion of quasi-tilings from [6]. Let X be a finite set and $\{A_i\}_{i=1}^n$ be finite subsets of X. Then we say that $\{A_1, A_2, \ldots, A_n\}$ are ϵ-disjoint if there exist subsets $\overline{A_i} \subseteq A_i$ such that

- For any $1 \leq i \leq n$, $\frac{|\overline{A_i}|}{|A_i|} \geq 1 - \epsilon$.

- If $i \neq j$ then $\overline{A_i} \cap \overline{A_j} = \emptyset$.

On the other hand, if $\{H_j\}_{j=1}^m$ are finite subsets of X, then we say that they α-cover X if

$$\frac{|X \cap (\cup_{j=1}^m H_j)|}{|X|} \geq \alpha \,.$$

Finally, we say that the collection $\{H_1, H_2, \ldots, H_m\}$ δ-evenly covers X if there exists some $M \in \mathbb{R}^+$ such that

- For any $p \in X$, $\sum_{j:p \in H_j}^m 1 \leq M$.

- $\sum_{j=1}^m |H_j| \geq (1 - \delta)M|X| \,.$

According to Lemma 4. [6], if $\{H_1, H_2, \ldots, H_m\}$ form a δ-even covering of X, then for any $0 < \epsilon < 1$ there exists an ϵ-disjoint subcollection of the H_j's that $\epsilon(1-\delta)$-covers X.

Now we define *tiles* for our S-edge colored graphs. Let G be a finitely generated group with a symmetric generator set S and let $1 \in F_1 \subseteq F_2 \subseteq \ldots, \cup_{n=1}^\infty F_n = G$ be a Folner-exhaustion. Let B be a finite graph as in the previous section with edge-colorings by the elements of S. Also, let L be a natural number. Let $\{F_{\alpha_1}, F_{\alpha_2}, \ldots F_{\alpha_n}\}$ be a finite collection of the Folner sets above such that for any $1 \leq i \leq n$, $F_{\alpha_i} \subset B_{\frac{1}{2}L}(1)$. Then for any $x \in Q_L^B$ and $1 \leq i \leq n$, $T_x(F_{\alpha_i})$ is the image of F_{α_i} under the unique colored isomorphism $\phi_L^x : B_L(1) \to B_L(x)$ mapping 1 to x. We call such a subset a tile of type F_{α_i} and say that x is the center of $T_x(F_{\alpha_i})$. A system of tiles ϵ-quasi tile $V(B)$ if they are ϵ-disjoint and form an $(1-\epsilon)$-cover. The following theorem is a version of Theorem 6. in [6].

Theorem 2 *For any $\epsilon > 0$, $n > 0$, there exist $L > 0$, $\delta > 0$ and a finite collection $\{F_{n_1}, F_{n_2}, \ldots, F_{n_s}\} \subset B_L(1)$ of the Folner sets, such that $n_i > n$ and if*

$$\frac{|Q_L^B|}{|V(B)|} > 1 - \delta$$

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then $V(B)$ can be ϵ-quasi-tiled by tiles of the form $T_x(F_{n_i})$, $x \in Q_L^B$, $1 \le i \le s$.

5   The inductional step

First of all fix a constant $\epsilon_1 < \frac{\epsilon}{100}$. Let us call a finite set $H \subset G$ a set of type (K, α), $K \in \mathbb{N}$, $\alpha \ge 0$ if

$$\frac{|B_K(H)|}{|H|} < 1 + \alpha. \tag{3}$$

Now let B be our S-edge colored finite graph and suppose that

$$\frac{|Q_L^B|}{|V(B)|} > 1 - \beta. \tag{4}$$

The exact values of β and L shall be given later. Assume that H is of type (K, α), where

$$H \subset B_{\frac{1}{100}L}(1) \quad \text{and} \quad K < \frac{L}{10}. \tag{5}$$

Now consider all tiles in B in the form $T_x(H)$, where $x \in Q_L^B$. Note that no vertices of B is covered by more than $|H|$ tiles. Indeed, if z is covered, then the $\frac{L}{2}$-neighborhood of z in B is colored isomorphic to the $\frac{L}{2}$-neighborhood of 1 in G. Hence if $z \in T_x(H)$, then $z \in Q_{\frac{L}{2}}^B$ and $x \in T_z(H^{-1})$. Summarizing these:

- For any $y \in V(B)$, $\sum_{x:y \in T_x(H)} 1 \le |H|$.

- $\sum_{x \in Q_L^B} |T_x(H)| = |Q_L^B||H| \ge (1 - \beta)|V(B)||H|$.

Consequently, the tiles $\{T_x(H)\}_{x \in Q_L^B}$ form a δ-even covering of $V(B)$, where $\delta = 1 - \beta$. Then by Lemma 4. of [6], there exists an ϵ_1-disjoint subcollection of tiles, $\cup_{x \in I} T_x(H)$ such that they form a $\epsilon_1(1 - \beta)$-covering of $V(B)$.

Now suppose that the number of vertices in $V(B)$ not covered by this subcollection above is greater than $\frac{\epsilon}{2}|V(B)|$. Let B_1 be the graph induced by the uncovered vertices. We would like to estimate the quotient: $\frac{|Q_K^{B_1}|}{|V(B)|}$. Note that if $y \in V(B_1)$ and

$$y \notin \cup_{x \in I}(T_x(B_K(H)) \backslash T_x(H))$$

then $y \in Q_K^{B_1}$. Hence by ϵ_1-disjointness,

$$\left| \cup_{x \in I} \left(T_x(B_K(H)) \backslash T_x(H) \right) \right| \le \alpha \sum_{x \in I} |H| \le \alpha(1 - \epsilon_1)^{-1} |V(B)|.$$

Hence

$$|Q_K^{B_1}| \ge |V(B_1)| - \alpha(1 - \epsilon_1)^{-1} |V(B)|,$$

that is

$$\frac{|Q_K^{B_1}|}{|V(B)|} \ge 1 - \beta_1, \tag{6}$$

where $\beta_1 = \alpha(1 - \epsilon_1)^{-1}\frac{2}{\epsilon}$. Also note that

$$\frac{\epsilon}{2}|V(B)| \le |V(B_1)| \le (1 - \epsilon_1(1 - \beta))|V(B)|.$$

6 The proof of Theorem 2

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of real numbers tending to zero and let $\{s_k\}_{k=1}^{\infty}$ be a sequence of real numbers tending to infinity, satisfying the following inequalities:

$$s_k \ge 1, s_{k+1} \ge 10s_k.$$

We call a subsequence of the Folner exhaustion $\{F_n\}_{n=1}^{\infty}$ an (α, s)-good subsequence if it satisfies the following conditions:

- $1 \in F_{n_1} \subset B_{s_1}(1) \subset F_{n_2} \subset B_{s_2}(1) \subset F_{n_3} \subset \dots$

- $F_{n_{i+1}}$ is of type $(100s_i, \alpha_i)$.

Obviously one can choose $\{s_k\}_{k=1}^{\infty}$ for any fixed $\{\alpha_k\}_{k=1}^{\infty}$ to have such (α, s)-good subsequences. Now let M be an integer such that

$$(1 - \frac{\epsilon_1}{2})^M < \frac{\epsilon}{100} \tag{7}$$

Also, pick $\beta > 0$ so that

$$\beta M < \frac{\epsilon}{100} \tag{8}$$

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And finally fix a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that

$$\alpha_i(1-\epsilon_1)^{-1}\frac{2}{\epsilon} < \beta. \tag{9}$$

Now let B a finite S-colored graph such that

$$\frac{Q_{100s_M}^B}{|V(B)|} > 1-\beta,$$

where β, M are as above. Then by the argument of the previous section we can $\epsilon_1(1-\beta)$-cover the vertices of B by ϵ_1-disjoint tiles of type F_{n_M}. If B_1 is the graph induced by the uncovered vertices of $V(B)$, by (6):

$$\frac{|Q_{s_M}^{B_1}|}{|V(B_1)|} > 1-\beta_1,$$

where $\beta_1 = \alpha_M(1-\epsilon_1)^{-1}\frac{2}{\epsilon}$. Now we can $\epsilon_1(1-\beta_1)$-cover $V(B_1)$ by tiles of type $F_{n_{M-1}}$. If B_2 denotes the graph induced by the uncovered part of $V(B_1)$ then

$$\frac{|Q_{s_{M-1}}^{B_2}|}{|V(B_2)|} > 1-\beta_2,$$

where $\beta_2 = \alpha_{M-1}(1-\epsilon_1)^{-1}\frac{2}{\epsilon}$.

We proceed inductively. In each step the new tiles are disjoint from all previous ones. Also,

$$V(B_i) \leq V(B_{i-1})(1-\frac{\epsilon_1}{2}).$$

Hence by our conditions, in at most M steps we obtain an ϵ-disjoint $(1-\epsilon)$-covering of $V(B)$. ∎

7   The proof of Theorem 1

Let G be a finitely generated amenable group, $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence converging to C_G. Let $\{T_n\}_{n=1}^{\infty}$ be the sequence of approximating operators as in Section 3.

Proposition 7.1 *For any pair $\delta, \epsilon > 0$ there exists $k_{\delta,\epsilon} > 0$ such that if $k \geq k_{\delta,\epsilon}$ then*

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \geq (\dim_G(\ker A) - \delta)(1-\epsilon).$$

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Proof. Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of G, such that all the F_n's have property $A(\epsilon, \delta, -)$ (see Lemma 2.1). Let $\{H_1, H_2, \ldots H_s\}$ be an ϵ-quasi-tiling of B_k by tiles from this Folner sequence. Such ϵ-quasi-tiling exists by Theorem 2 if k is large enough. For $1 \leq i \leq s$ let $K_i \subset H_i$ be a subset such that

- $\frac{|K_i|}{|H_i|} > 1 - \epsilon$.

- $K_i \cap K_j = \emptyset$ if $i \neq j$.

Since the F_n's have property $A(\epsilon, \delta, -)$ there exist subspaces $V_i \subset (l^2(B_n))^d$ such that

- If $f \in V_i$ then $\operatorname{supp} f \subseteq K_i$.

- $f \in \ker T_k$.

- $\frac{\dim_{\mathbb{C}} V_i}{|H_i|} \geq \dim_G(\ker A) - \delta$.

Now consider the subspace $\oplus_{i=1}^s V_i \subseteq \ker T_k$. Then

$$\dim_{\mathbb{C}}(\oplus_{i=1}^s V_i) \geq (\sum_{i=1}^s |H_i|)(\dim_G(\ker A) - \delta) \geq (1 - \epsilon)|V(B_k)|(\dim_G(\ker A) - \delta).$$

That is

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \geq (\dim_G(\ker A) - \delta)(1 - \epsilon). \qquad \blacksquare$$

Proposition 7.2 *For any pair $\delta, \epsilon > 0$ there exists $m_{\delta, \epsilon} > 0$ such that if $k \geq m_{\delta, \epsilon}$ then*

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \leq (1 - \epsilon)^{-1}(\dim_G(\ker A) + \delta) + \epsilon.$$

Proof. Again let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of G, such that all the F_n's have property $A(\delta, +)$ (see Lemma 2.1). Consider the ϵ-quasi-tilings of the previous proposition. Now let $W_i \subset l^2(H_i)$ be the restriction of $\ker T_k$ onto H_i. By our assumption,

$$\dim_G(\ker T_k) \leq |H_i|(\dim_G(\ker A) + \delta).$$

Since $\{H_i\}_{i=1}^s$ form an ϵ-covering

$$\dim_{\mathbb{C}}(\ker T_k) \leq \epsilon|V(B_k)| + \sum_{i=1}^s |H_i|(\dim_G(\ker A) + \delta).$$

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Note that by ϵ-disjointness

$$\sum_{i=1}^{s} |H_i| \leq (1 - \epsilon)^{-1} |V(B_k)| \,.$$

Thus

$$\frac{\dim_{\mathbb{C}} (\ker T_k)}{|V(B_k)|} \leq (1 - \epsilon)^{-1} (\dim_G(\ker A) + \delta) + \epsilon \,. \qquad \blacksquare$$

Clearly, Propositions 7.1 and 7.2 imply Theorem 1. $\qquad \blacksquare$

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